## Friday, October 9, 2015

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### Problem 5

*Problem.* A force of 5 pounds compresses a 15-inch spring a total of 3 inches. How much work is done in compressing the spring 7 inches?

*Solution.* The force is $F(x) = kx$, for some $k$. We are given that $F(3) = 5$, so it follows that $k = \frac{5}{3}$. The work done in compressing the spring 7 inches is

$$\begin{aligned} W &= \int_0^7 \frac{5}{3}x\,dx \\ &= \frac{5}{3}\left[\frac{1}{2}x^2\right]_0^7 \\ &= \frac{5}{6}\cdot 7^2 \\ &= \frac{245}{6}\text{ in-lb.} \end{aligned}$$

### Problem 7

*Problem.* A force of 20 pounds stretches a spring 9 inches in an exercise machine. Find the work done in stretching the spring 1 foot from its natural position.

*Solution.* We are given that $F(9) = 20$, so it follows that $k = \frac{20}{9}$. The work done in stretching the spring 12 inches is

$$\begin{aligned} W &= \int_0^{12} \frac{20}{9}x\,dx \\ &= \frac{20}{9}\left[\frac{1}{2}x^2\right]_0^{12} \\ &= \frac{10}{9}\cdot 12^2 \\ &= 160\text{ in-lb.} \end{aligned}$$

### Problem 11

*Problem.* Determine the work done in propelling a five-ton satellite to a height of (a) 100 miles above the Earth and (b) 300 miles above the Earth.

*Solution.* The weight of the satellite is 10,000 lbs. The force is

$$F(x) = \frac{GmM}{x^2}$$

where $x$ is the distance from the satellite to the center of the earth. So

$$\begin{aligned} F(4000) &= \frac{GmM}{4000^2} \\ &= \frac{GmM}{16 \times 10^6} \\ &= 10,000. \end{aligned}$$

Therefore, $GmM = 16 \times 10^{10}$ and we have

$$F(x) = \frac{16 \times 10^{10}}{x^2}.$$

(a) The work done is

$$\begin{aligned} W &= \int_{4000}^{4100} \frac{16 \times 10^{10}}{x^2}\, dx \\ &= 16 \times 10^{10} \left[ -\frac{1}{x} \right]_{4000}^{4100} \\ &= 16 \times 10^{10} \left( -\frac{1}{4100} + \frac{1}{4000} \right) \\ &= 16 \times 10^{10} \left( \frac{100}{(4000)(4100)} \right) \\ &= \frac{4 \times 10^7}{41} \text{ mi-lbs} \end{aligned}$$

(b) The work done is

$$\begin{aligned} W &= \int_{4000}^{4300} \frac{16 \times 10^{10}}{x^2}\, dx \\ &= 16 \times 10^{10} \left[ -\frac{1}{x} \right]_{4000}^{4300} \\ &= 16 \times 10^{10} \left( -\frac{1}{4300} + \frac{1}{4000} \right) \\ &= 16 \times 10^{10} \left( \frac{300}{(4000)(4300)} \right) \\ &= \frac{12 \times 10^7}{43} \text{ mi-lbs} \end{aligned}$$

## Problem 12

*Problem.* Write the work $W$ of the propulsion system as a function of the height $h$ of the satellite above Earth. Find the limit of $W$ as $h$ approaches infinity.

*Solution.* In the solution to Problem 11, replace 4100 (or 4300) with $h$ and redo the integration.

$$\begin{aligned} W &= \int_{4000}^{h} \frac{16 \times 10^{10}}{x^2}\, dx \\ &= 16 \times 10^{10} \left[ -\frac{1}{x} \right]_{4000}^{h} \\ &= 16 \times 10^{10} \left( \frac{1}{4000} - \frac{1}{h} \right). \end{aligned}$$

Now take the limit as $h \to \infty$.

$$\begin{aligned} \lim_{h \to \infty} 16 \times 10^{10} \left( \frac{1}{4000} - \frac{1}{h} \right) &= 16 \times 10^{10} \left( \frac{1}{4000} \right) \\ &= 4 \times 10^{7} \text{ ft-lb.} \end{aligned}$$

## Problem 15

*Problem.* A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water. The water weighs 62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty (a) half of the tank and (b) all of the tank?

*Solution.* Let the $x$-axis be vertical, measuring the vertical distance that the water is raised, and let 0 be at the base of the tank, 4 at the top of the tank. In either case, (a) or (b), a horizontal slice of $(4 \times 5 \times \Delta x) \times 62.4$ lbs. So the work done in lifting that slice is $W_i = 1248\Delta x$ ft-lb.

(a) The range of $x$ is from 2 to 4, so the total work done is

$$\begin{aligned}
W &= \int_2^4 1248(4-x)\,dx \\
&= 1248\left[4x - \frac{1}{2}x^2\right]_2^4 \\
&= 1248\left(\left(16 - \frac{1}{2}\cdot 16\right) - \left(8 - \frac{1}{2}\cdot 4\right)\right) \\
&= 1248(8-6) \\
&= 2496 \text{ ft-lb.}
\end{aligned}$$

(b) The range of $x$ is from 0 to 4, so the total work done is

$$\begin{aligned}
W &= \int_0^4 1248(4-x)\,dx \\
&= 1248\left[4x - \frac{1}{2}x^2\right]_0^4 \\
&= 1248\left(16 - \frac{1}{2}\cdot 16\right) \\
&= 1248(8) \\
&= 9984 \text{ ft-lb.}
\end{aligned}$$

**Problem 17**

*Problem.* A cylindrical water tank 4 meters high with a radius of 2 meters is buried so that the top of the tank is 1 meter below ground level. How much word is done in pumping a fulll tank of water up to ground level?

*Solution.* Let the 0-point on the $x$-axis be at the bottom of the tank, so the level of the water ranges from $x = 0$ to $x = 4$. (It would just as well to let 0 be ground level and let $x$ range from $x = -5$ to $x - 0$.) Then a slice of water at level $x$ is raised $5 - x$ meters and it weighs $(\pi \cdot 2^2 \cdot \Delta x) \times 9800 = 39200\pi\Delta x$ N.

The work done is

$$\begin{aligned}
W &= \int_0^4 39200\pi(5-x)\,dx \\
&= 39200\pi\left[5x - \frac{1}{2}x^2\right]_0^4 \\
&= 39200\pi\,(20-8) \\
&= 470400\pi \text{ N-m}
\end{aligned}$$

**Problem 19**

*Problem.* An open tank has the shape of a right circular cone. The tank is 8 feet across the top and 6 feet high. How much work is done in emptying the tank by pumping the water over the top edge?

*Solution.* (I'll use $y$ because the book uses $y$.) The radius-to-height ratio is 6 :: 8, or 3/4. So the radius of a cross-section at level $y$ is $3y/4$. A cross-section at level $y$ has volume $\pi\left(\frac{3y}{4}\right)^2\Delta y$. That cross-section is lifted a distance $6-y$, so the work done is

$$\begin{aligned}
W &= \int_0^6 \pi\left(\frac{3y}{4}\right)^2(6-y)\rho\,dy \\
&= \frac{9\pi\rho}{16}\int_0^6 y^2(6-y)\,dy \\
&= \frac{9\pi\rho}{16}\int_0^6 (6y^2-y^3)\,dy \\
&= \frac{9\pi\rho}{16}\left[2y^3 - \frac{1}{4}y^4\right]_0^6 \\
&= \frac{9\pi\rho}{16}(432-324) \\
&= \frac{243\pi\rho}{4} \text{ ft-lb.}
\end{aligned}$$

**Problem 20**

*Problem.* Water is pumped in through the bottom of the tank in Exercise 19. How much work is done to fill the tank

(a) to a depth of 2 feet?

(b) from a depth of 4 feet to a depth of 6 feet?

*Solution.* The integral is the same as in the Exercise 19, except that the distance lifted is $y$, not $6 - y$.

(a)

$$\begin{aligned} W &= \int_0^2 \pi \left(\frac{3y}{4}\right)^2 \cdot y\rho \, dy \\ &= \frac{9\pi\rho}{16} \int_0^2 y^3 \, dy \\ &= \frac{9\pi\rho}{16} \left[\frac{1}{4} y^4\right]_0^2 \\ &= \frac{9\pi\rho}{16}(4) \\ &= \frac{9\pi\rho}{4} \text{ ft-lb.} \end{aligned}$$

(b) This is the same as part (a) except the limits of integration are different.

$$\begin{aligned} W &= \int_4^6 \pi \left(\frac{3y}{4}\right)^2 \cdot y\rho \, dy \\ &= \frac{9\pi\rho}{16} \int_4^6 y^3 \, dy \\ &= \frac{9\pi\rho}{16} \left[\frac{1}{4} y^4\right]_4^6 \\ &= \frac{9\pi\rho}{16}(324 - 64) \\ &= \frac{585\pi\rho}{4} \text{ ft-lb.} \end{aligned}$$

**Problem 21**

*Problem.* A hemispherical tank of radius 6 feet is positioned so that its base is circular. How much work is required to fill the tank with water through a hole in the base when the water source is at the base?

*Solution.* An equation of a vertical cross-section through the middle of the tank is $y = \sqrt{36 - x^2}$ (a semicircle of radius 6). Now consider a horizontal cross-section through the tank at height $y$ above the ground. The cross-section is circular with radius $\sqrt{36-y^2}$. This cross-section is lifted a distance $y$, so the work done is

$$\begin{aligned} W &= \int_0^6 \pi(36-y^2)y\rho\, dy \\ &= \pi\rho \int_0^6 (36y - y^3)\, dy \\ &= \pi\rho \left[18y^2 - \frac{1}{4}y^4\right]_0^6 \\ &= \pi\rho\,(648 - 324) \\ &= 324\pi\rho \text{ ft-lb.} \end{aligned}$$

**Problem 25**

*Problem.* Consider a 20-foot chain that weighs 3 pounds per foot hanging from a winch 20 feet above ground level. Find the work done by the winch in winding up the entire chain.

*Solution.* When the bottom of the chain is at a height of $x$, the remaining length is $20-x$ feet and the weight is $3(20-x)$ pounds. To lift that length a distance $\Delta x$ requires $3(20-x)\Delta x$ ft-lb of work. The total work done is

$$\begin{aligned} W &= \int_0^{20} 3(20-x)\, dx \\ &= 3\left[20x - \frac{1}{2}x^2\right]_0^{20} \\ &= 3(400-200) \\ &= 600 \text{ ft-lb.} \end{aligned}$$

**Problem 26**

*Problem.* Consider a 20-foot chain that weighs 3 pounds per foot hanging from a winch 20 feet above ground level. Find the work done by the winch in winding up one-third of the chain.

*Solution.* The is the same as the previous problem, except that the limits of integration are from $x = 0$ to $x = 20/3$.

$$\begin{aligned} W &= \int_0^{20/3} 3(20 - x)\, dx \\ &= 3\left[20x - \frac{1}{2}x^2\right]_0^{20/3} \\ &= 3(\frac{400}{3} - \frac{400}{18}) \\ &= \frac{1000}{3} \text{ ft-lb.} \end{aligned}$$